

AFCRC-TN-60-111



**NEW YORK UNIVERSITY**

Institute of Mathematical Sciences

Division of Electromagnetic Research

25 Waverly Place, New York 3, N. Y.

RESEARCH REPORT No. EM-150

# Asymptotic Solution of Systems of Linear Ordinary Differential Equations with Discontinuous Coefficients

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Contract No. AF 19(604)5238

DECEMBER, 1959



NEW YORK UNIVERSITY  
Institute of Mathematical Sciences  
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DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS  
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The research reported in this document has been sponsored by the Electronics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract No. AF 19(604)5238.

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### Abstract

A system of linear first order ordinary differential equations is considered in which the coefficient matrix depends upon a parameter  $k$ . The asymptotic form of the solutions of this system are determined for large values of  $k$  when the coefficient matrix or some of its derivatives have finite jump discontinuities. The asymptotic form obtained contains terms beyond those previously known. These new terms have discontinuous coefficients, whereas the previously known terms are continuous.

The results are applied to the one dimensional propagation of time-harmonic waves. It is shown that in this case the reflection coefficient is proportional to  $k^{-J}$  if the  $J$ -th derivative of the index of refraction is discontinuous and the lower order derivatives are continuous. Here  $k = 2\pi/\lambda$  is the propagation constant and  $\lambda$  the wavelength in a medium of unit refractive index. This result is to be contrasted with the fact that the reflection coefficient vanishes faster than any negative power of  $k$  if all derivatives of the index of refraction are continuous.

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## 1. Introduction

Let us consider the vector solution  $u(x,k)$  of the system of  $N$  first order linear ordinary differential equations

$$(1) \quad \frac{du}{dx} = kA(x,k)u$$

In this system the coefficient matrix  $kA$  depends upon a real parameter  $k$ . Asymptotic expansions of  $u$  in powers of  $k^{-1}$ , which are valid as  $k \rightarrow \infty$ , are well known. They hold provided that  $A$  has continuous derivatives of all orders and satisfies certain other conditions. But if some derivative of  $A$  is discontinuous, then only a finite number of terms in the expansion of  $u$  are known. We consider the problem of finding additional terms when  $A$  and its derivatives have finite jump discontinuities. We shall show how to solve this problem, and we shall find explicitly the next term beyond those previously known. This term is discontinuous, whereas the preceding terms are all continuous. This fact is significant in various applications.

One important application of our result is to one dimensional propagation of time harmonic waves in an inhomogeneous medium. Such a wave motion  $w$  satisfies the reduced wave equation

$$(2) \quad \frac{d^2 w}{dx^2} + k^2 n^2(x)w = 0.$$

Here  $n(x)$  denotes the refractive index of the medium and  $k$  denotes the propagation constant in a medium with  $n = 1$ . It follows from known results that the reflection coefficient  $R(k)$  determined from (2) is asymptotically zero to all orders in  $k^{-1}$ , provided that  $n$  has continuous derivatives of all orders. However, if the  $J$ -th derivative of  $n$  has a finite jump discontinuity, and the

lower order derivatives are continuous, our result shows that  $R(k)$  is of the order  $k^{-J}$ . This essential difference between the infinitely and finitely differentiable cases has been observed by S.A. Schelkunoff<sup>[1]</sup> and J. Feinstein<sup>[2]</sup>. Equation (2) is treated in section 4.

## 2. Preliminaries

We wish to determine the asymptotic behavior, for large values of  $k$ , of the solutions of (1) in an interval  $I$  of the real  $x$  axis. We assume that within  $I$  the matrix  $A$  has the asymptotic expansion

$$(3) \quad A(x, k) \sim \sum_{j=0}^{\infty} A_j(x) k^{-j}.$$

We also assume that  $A_0(x)$  has  $N$  eigenvalues  $\lambda_1(x), \dots, \lambda_N(x)$  which are distinct in  $I$ . Then there exists a non singular matrix  $R(x)$  which diagonalizes  $A_0(x)$  in  $I$ . We define the diagonal matrix  $\Lambda(x)$  by the equation

$$(4) \quad \Lambda(x) = R^{-1} A_0 R \equiv \text{diag} [\lambda_1(x), \dots, \lambda_N(x)].$$

The matrix  $R$  can be so chosen that, like  $\Lambda$ , it is independent of  $k$  and possesses as many continuous derivatives as does  $A_0(x)$ .

With the aid of  $R$  we define the vector  $v(x, k)$  by the equation

$$(5) \quad u(x, k) = R(x) v(x, k).$$



Then from (1), it follows that  $v$  satisfies the equation

$$(6) \quad \frac{dv}{dx} = kB(x,k)v.$$

Here the matrix  $B$  is defined by

$$(7) \quad B(x,k) = R^{-1}AR - k^{-1}R^{-1}dR/dx.$$

It has the asymptotic expansion

$$(8) \quad B(x,k) \sim \sum_{j=0}^{\infty} B_j(x)k^{-j}.$$

The  $B_j$  are given by

$$(9) \quad B_0 = A, \quad B_1 = R^{-1}A_1R - R^{-1}dR/dx, \quad B_j = R^{-1}A_jR, \quad j = 2, 3, \dots$$

We shall determine the asymptotic behavior of the solutions  $v$  of (6). Then the behavior of the solutions  $u$  of (1) will be found by means of (5). The advantage of dealing with (6) rather than (1) is that  $B_0$  is diagonal while  $A_0$  is not necessarily diagonal.

Let us tentatively assume that (6) has  $N$  linearly independent solutions  $v_i(x,k)$  with asymptotic expansions of the form

$$(10) \quad v_i(x,k) \sim \exp \left[ k \int_{x_0}^x \lambda_i(x) dx \right] \left( \sum_{s=0}^{\infty} v_i^s(x) k^{-s} \right) \quad i = 1, \dots, N.$$

Let us also suppose that the expansion of  $dv/dx$  is obtained by termwise differentiation of (10). Then we can insert (10) and (8) into (6) and equate to zero the coefficient of each power of  $k^{-1}$ . In this way we obtain the following set of equations:

$$(11) \quad M_i v_i^s = dv_i^{s-1}/dx - \sum_{j=0}^{s-1} B_{s-j} v_i^j \quad s = 0, 1, \dots$$

Quantities with negative subscripts or superscripts in (11) and elsewhere are defined to be zero, and  $M_i$  is defined by

$$(12) \quad M_i = B_0 - \lambda_i I = \text{diag}[\lambda_1 - \lambda_i, \dots, \lambda_N - \lambda_i].$$

$M_i$  is diagonal with a zero in the  $(i, i)$  position and it is therefore a singular matrix.

If we set  $s = 0$  in (11) we see that the right side vanishes. Therefore the only non zero component of  $v_i^0$  is the  $i^{\text{th}}$  one. Then the solvability condition for (11) with  $s \neq 0$  is simply that the  $i^{\text{th}}$  component of the right side vanish. Upon replacing  $s$  by  $s + 1$ , this condition can be written in the form

$$(13) \quad dv_{ii}^s/dx - (B_1 v_i^s)_i = \sum_{j=0}^{s-1} (B_{s+1-j} v_i^j)_i \quad s = 0, 1, \dots$$

The second subscript indicates the component of the vector. Equation (13) is a first order ordinary differential equation for  $v_{ii}^s$ . From (11) all other components of  $v_i^s$  can be determined in terms of the  $v_i^j$  with  $j < s$ . Then  $v_{ii}^s$  can be found from (13). Thus by starting with  $s = 0$ , we can determine successively all the  $v_i^s$ .

By keeping track of the differentiation which is performed in (11) at each step of this process, we can determine the continuity properties of the derivatives of the  $v_1^s$  in terms of those of the  $B_j$  and therefore of the  $A_j$ . In this way we arrive at the following lemma, which is due to P.D. Tamarkin [3].

Lemma If for some  $J_0 \geq 0$  and all  $j \leq J_0$ , each coefficient  $A_j$  has  $J_0 - j$  continuous derivatives, then so does  $v_i^j$  for  $i = 1, \dots, N$ .

This lemma can be verified readily by induction, with the aid of (11) and (13). Tamarkin has also proved the following theorem.

Theorem I Suppose that in an interval  $I$  of the  $x$  axis  $A(x, k)$  has the asymptotic form

$$(14) \quad A(x, k) = \sum_{j=0}^{J_0} A_j(x) k^{-j} + o(k^{-J_0}).$$

Let the  $A_j$  satisfy the hypothesis of the lemma and let  $A_0(x)$  have distinct eigenvalues  $\lambda_i(x)$ ,  $i = 1, \dots, N$  in  $I$ . Let  $u_i^s(x) = R v_i^s$  where  $v_i^s$  are the vectors constructed as described above and  $R$  is a matrix which diagonalizes  $A_0$  and has as many continuous derivatives as does  $A_0$ . Then there exist  $N$  linearly independent solutions of (1),  $u_i(x, k)$ ,  $i = 1, \dots, N$ , such that in  $I$

$$(15) \quad u_i(x, k) = \exp \left[ k \int_{x_0}^x \lambda_i(x) dx \right] \left( \sum_{j=0}^{J_0} u_i^s(x) k^{-s} + o(k^{-J_0}) \right)$$

The coefficient  $u_i^s(x)$  has  $J - s$  continuous derivatives.

### 3. The discontinuous case

Let us now suppose that  $J$  is the largest value of  $J_0$  for which the hypothesis of the lemma is satisfied. Then for at least one value of  $j$  in the range  $0 \leq j \leq J + 1$ , the coefficient  $A_j$  has a discontinuous  $(J + 1 - j)$ th

derivative. Let us suppose that this discontinuity occurs at  $x_0$ . Then we may choose  $I$  so small that for each  $j$  in the range  $0 \leq j \leq J + 1$ , the  $(J + 1 - j)$ -th derivative of  $A_j$  is continuous in  $I$ , except perhaps at  $x_0$ . We further assume that the only discontinuities of any of these derivatives are finite jump discontinuities at  $x_0$ . From the definition (9) of the  $B_j$  it follows that the properties just described pertain to them also.

Let us now consider the solutions  $v(x, k)$  of (6) in which the coefficient matrix is  $B(x, k)$ . With the assumption just made on the  $B_j$ , the hypotheses of the lemma and of Theorem I are satisfied by  $B$  with  $J_0 = J + 1$ , in each of the two subintervals  $I^-$  and  $I^+$  into which  $x_0$  divides  $I$ . Therefore there are  $N$  linearly independent solutions  $v_i$  of (6) which have the asymptotic form (15) in  $I^-$  with  $J_0 = J + 1$ , and  $N$  others  $w_i$  which have that form in  $I^+$ . We wish to determine the asymptotic form of, say, the  $v_i$  in  $I^+$ . To do this we first choose  $w_i$  so that its asymptotic form coincides with that of  $v_i$  throughout  $I$ , up to and including terms of order  $k^{-J}$ . That this is possible follows from Theorem I since we may choose the coefficients  $w_i^j = v_i^j$  for  $j = 0, \dots, J$  and then determine  $w_i^{J+1}$  in  $I^+$  by the construction of the previous section. Since the component  $w_{ii}^{J+1}$  is obtained by integrating a first order differential equation, its value can be arbitrarily prescribed at one point. Therefore for later convenience we prescribe

$$(16) \quad w_{ii}^{J+1}(x_0) = v_{ii}^{J+1}(x_0).$$

Then the theorem guarantees the existence of a solution  $w_i$  with these coefficients.

By the linear independence of the  $w_i$  we may write

$$(17) \quad v_i(x, k) = \sum_{s=0}^N c_{is}(k) w_s(x, k).$$

Since the coefficients in the asymptotic form of  $w_i$  coincide with those of  $v_i$  through terms in  $k^{-J}$  it follows that the constant  $c_{is}$  differs from the Kronecker  $\delta_{is}$  by terms of order  $k^{-J-1}$ . Therefore there are constants  $\gamma_{is}$  such that

$$(18) \quad c_{is}(k) = \delta_{is} + \gamma_{is} k^{-J-1} + o(k^{-J-1}).$$

When (18) and the asymptotic form of  $w_s$  for  $x$  in  $I^+$  are inserted into (17), the following asymptotic form of  $v_i$  is obtained in  $I^+$ :

$$(19) \quad v_i(x, k) = \exp \left[ k \int_{x_0}^x \lambda_i(x) dx \right] \sum_{s=0}^{J+1} w_i^s(x) k^{-s} + \sum_{t=1}^N \exp \left[ k \int_{x_0}^x \lambda_t(x) dx \right] \\ \times \gamma_{it} w_t^0(x) \left[ k^{-J-1} + o(k^{-J-1}) \right], \quad x \geq x_0.$$

Upon comparing (19) with (15) we see that whereas the asymptotic form of  $v_i$  contains only one of the exponentials in  $I^-$ , it contains all of them in  $I^+$ . To determine  $c_{is}$  we set  $x = x_0$  in (17) and solve the resulting linear equations. They have a unique solution because the  $w_s$  are linearly independent. In this way  $c_{is}$  is given in terms of the  $v_i(x_0, k)$  and  $w_i(x_0, k)$ . Since  $x_0$  belongs to both  $I^-$  and  $I^+$ , both  $v_i$  and  $w_i$  have asymptotic forms of the type (15) at  $x_0$ , with  $J_0 = J + 1$ . By inserting them into the expression for  $c_{is}$  the asymptotic form of  $c_{is}$  is determined. In particular, to determine  $\gamma_{it}$  we set  $x = x_0$  in (19) and use the asymptotic form of  $v_i$  given by (15), which applies at  $x = x_0$ . Then the coefficients of  $k^{-J-1}$  in (19) yield

$$(20) \quad v_i^{J+1}(x_0) = w_i^{J+1}(x_0) + \sum_{t=1}^N \gamma_{it} w_t^0(x_0), \quad i = 1, \dots, N.$$

To solve (20) we recall that the only nonzero component of  $w_t^0$  is  $w_{tt}^0$  because  $w_t^0$  is an eigenvector of the diagonal matrix  $B_0$ . Therefore only one  $\gamma_{it}$  occurs in each component of the right member of (20). Solving (20) yields

$$(21) \quad \gamma_{it} = \frac{v_{it}^{J+1}(x_0) - w_{it}^{J+1}(x_0)}{w_{tt}^0(x_0)}.$$

From (16) and (21) we see that

$$(22) \quad \gamma_{ii} = 0.$$

We shall now compute the right member of (21) in terms of the coefficients  $B_j$ , and thus express the  $\gamma_{it}$  in terms of them. For this purpose it is convenient to introduce the special jump notation

$$(23) \quad [v_i^j] = w_i^j(x_0+) - v_i^j(x_0-).$$

When applied to  $B_j$  or  $M_j$ , the jump notation will have its usual meaning

$$(24) \quad [B_j] = B_j(x_0+) - B_j(x_0-).$$

Let us now consider (11) with  $s = J + 1$ . We write (11) for  $v_i^{J+1}$  at  $x_0-$  and for  $w_i^{J+1}$  at  $x_0+$  and subtract the two equations. By hypothesis  $M_i$  and  $B_{J+1-j}$  with  $j = 1, \dots, J$  are continuous. By construction  $v_i^j(x_0) = w_i^j(x_0)$  for  $j = 0, \dots, J$ . Thus we obtain

$$(25) \quad M_j[v_i^{J+1}] = [dv_i^J/dx] - [B_{J+1}]v_i^0.$$

The left side of (25) contains the jumps we need in (21) to compute  $\gamma_{it}$ . To determine them we must eliminate the derivative from (25). To this end we now write (11) with  $s = J$  and differentiate it with respect to  $x$  obtaining

$$(26) \quad \frac{dM_i}{dv} v_i^J + M_i \frac{dv_i^J}{dx} = \frac{d^2 v_i^{J-1}}{dx^2} - \sum_{j=0}^{J-1} \frac{dB_{J-j}}{dx} v_i^j + B_{J-j} \frac{dv_i^j}{dx} .$$

By construction  $v_i^j(x_0) = w_i^j(x_0)$  for  $j = 0, \dots, J$  and by hypothesis  $dM_i/dx$  is continuous if  $J > 0$ . Also by hypothesis  $dB_{J-j}/dx$  is continuous for  $j = 1, \dots, J-1$  and  $B_{J-j}$  for  $j = 0, \dots, J-1$ . By the lemma  $dv_i^j/dx$  is continuous for  $j = 0, \dots, J-1$ . Thus upon subtracting (26) at  $x_0^-$  from the corresponding equation for  $w$  at  $x_0^+$ , we obtain

$$(27) \quad M_i \left[ \frac{dv_i^J}{dx} \right] = \left[ \frac{d^2 v_i^{J-1}}{dx^2} \right] - \left[ \frac{dB_J}{dx} \right] v_i^0 .$$

We now multiply (25) by  $M_i$  and combine it with (27) to yield

$$(28) \quad (M_i)^2 [v_i^{J+1}] = \left[ \frac{d^2 v_i^{J-1}}{dx^2} \right] - \left( \left[ \frac{dB_J}{dx} \right] + M_i [B_{J+1}] \right) v_i^0 .$$

By continuing in the same way we obtain for any  $m \leq J + 1$

$$(29) \quad (M_i)^m [v_i^{J+1}] = \left[ \frac{d^m v_i^{J-m+1}}{dx^m} \right] - \sum_{r=0}^{m-1} (M_i)^r \left[ \frac{d^{m-r-1} B_{J-m+r+2}}{dx^{m-r-1}} \right] v_i^0 .$$

Specializing (29) to  $m = J + 1$  yields

$$(30) \quad (M_i)^{J+1} [v_i^{J+1}] = \left[ \frac{d^{J+1} v_i^0}{dx^{J+1}} \right] - \sum_{r=0}^J (M_i)^r \left[ \frac{d^{J-r} B_{r+1}}{dx^{J-r}} \right] v_i^0 .$$

To eliminate the derivative of  $v_i^0$  we multiply (30) by  $M_i$ . Since the only non-zero element of  $v_i^0$  is  $v_{ii}^0$ , and since the  $ii$  element of  $M_i$  is zero, the product  $M_i [d^{J+1} v_i^0 / dx^{J+1}] = 0$ . Then (30) becomes

$$(31) \quad (M_i)^{J+2} [v_i^{J+1}] = - \sum_{r=0}^J (M_i)^{r+1} \left[ \frac{d^{J-r} B_{r+1}}{dx^{J-r}} \right] v_i^o, \quad x = x_o.$$

Since  $M_i$  is diagonal with a zero in the  $ii$  position (31) can be solved for the components  $[v_{it}^{J+1}]$  with  $t \neq i$ . The result is

$$(32) \quad [v_{it}^{J+1}] = - \sum_{r=0}^J (\lambda_t - \lambda_i)^{r-J-1} \left[ \frac{d^{J-r} B_{r+1, ti}}{dx^{J-r}} \right] v_{ii}^o, \quad i \neq t.$$

Finally, (32) and (21) yield

$$(33) \quad \gamma_{it} = - \frac{v_{ii}^o}{v_{tt}^o} \sum_{r=0}^J (\lambda_t - \lambda_i)^{r-J-1} \left[ \frac{d^{J-r} B_{r+1, ti}}{dx^{J-r}} \right], \quad x = x_o, \quad i \neq t.$$

This result together with (22) completely determine  $\gamma_{it}$ . Thus the asymptotic form (19) of  $v_i$  in  $I^+$  is determined. Then the asymptotic form of  $u_i$ , the solution of (1), is given by multiplying the asymptotic form of  $v_i$  by the matrix  $R$ , according to (5).

We may summarize our results as

Theorem II Suppose that in an interval  $I$  of the  $x$  axis,

$A(x, k)$  has the asymptotic form

$$(34) \quad A(x, k) = \sum_{j=0}^{J+1} A_j(x) k^{-j} + o(k^{-J-1}), \quad J \geq 0.$$

Let  $A_o(x)$  have  $N$  distinct eigenvalues  $\lambda_i(x)$ ,  $i=1, \dots, N$ , in  $I$ . Furthermore, let  $A_j$  have  $J-j$  continuous derivatives in  $I$  for  $j=0, \dots, J$ . Suppose that the only discon-



tinuities in I of the derivatives of order  $(J + 1 - j)$ ,  
 $j = 0, \dots, J+1$  are finite jumps at  $x_0$ . Let  $u_i^s(x) = R(x)v_i^s(x)$   
and  $\tilde{u}_i^s(x) = R w_i^s(x)$ ,  $i = 1, \dots, N$  and  $s = 0, \dots, J+1$ , where  
 $v_i^s$  and  $w_i^s$  are the vectors constructed as described above  
and  $R$  is a matrix which diagonalizes  $A_0$  and has as many  
continuous derivatives as does  $A_0$ . Let the constants  $\gamma_{it}$   
be given by (33) for  $i \neq t$ , and  $\gamma_{ii} = 0$ . The coefficients  
 $u_i^s$  and  $\tilde{u}_i^s$  have  $J-s$  continuous derivatives and at  $x_0$  these  
derivatives of  $u_i^s$  equal those of  $\tilde{u}_i^s$  for  $s = 0, \dots, J$ . Then  
there exist  $N$  linearly independent solutions of (1),  
 $u_i(x, k)$ ,  $i = 1, \dots, N$  with the asymptotic forms

$$(35) \quad u_i(x, k) = \exp \left[ k \int_{x_0}^x \lambda_i(x) dx \right] \left[ \sum_{s=0}^{J+1} u_i^s(x) k^{-s} + o(k^{-J-1}) \right] \quad x \leq x_0$$

$$(36) \quad u_i(x, k) = \exp \left[ k \int_{x_0}^x \lambda_i(x) dx \right] \left[ \sum_{s=0}^{J+1} \tilde{u}_i^s(x) k^{-s} + o(k^{-J-1}) \right] \\ + \sum_{t=1}^N \exp \left[ k \int_{x_0}^x \lambda_t(x) dx \right] \left[ \gamma_{it} \tilde{u}_t^0(x) k^{-J-1} + o(k^{-J-1}) \right] \quad x \geq x_0$$

#### 4. Application

Let us now consider the asymptotic form of the solutions of (2) when there  
is a finite jump in the  $(J+1)$ -st derivative of the coefficient  $n^2(x)$ . This  
coefficient is the square of the refractive index in wave propagation. We first  
introduce the two component vector  $u$  defined by

$$(37) \quad u = \begin{pmatrix} kw \\ w' \end{pmatrix}$$

Then it follows from (2) that  $u$  satisfies (1) with  $A$  given by

$$(38) \quad A(x) = \begin{pmatrix} 0 & 1 \\ -n^2(x) & 0 \end{pmatrix}$$

We see that  $A$  is of the form (3) with  $A_0 = A$ ,  $A_j \equiv 0$  for  $j \geq 1$ . The eigenvalues of  $A_0(x)$  are  $\lambda_1 = in(x)$  and  $\lambda_2 = -in(x)$ , which are distinct if  $n \neq 0$ . If the first  $J$  derivatives of  $n$  are continuous, while the  $(J+1)$ -st derivative has a jump at  $x_0$ , the same is true of  $A_0$  provided  $n(x_0) \neq 0$ . Thus the hypotheses of Theorem II are satisfied and therefore (1) has two linearly independent solutions with the asymptotic forms (35) and (36).

If  $n(x) > 0$  we may interpret the factors

$$\exp \left[ ik \int_{x_0}^x n dx \right] \text{ and } \exp \left[ -ik \int_{x_0}^x n dx \right]$$

as representing waves traveling to the right and left respectively. Then the solution  $u_2(x)$  represents a wave traveling to the left in the region  $x \leq x_0$ , according to (35). However, by (36) it represents both left and right traveling waves in the region  $x \geq x_0$ . We may interpret this solution as corresponding to an incident wave traveling to the left in the region  $x > x_0$  and being partly transmitted and partly reflected at  $x = x_0$ . The reflected wave is of the order  $k^{-J-1}$  with respect to the incident wave, as we see from (36). Equation (36) also shows that the first component of the reflected wave, which is  $kw$  according to (37), is proportional to  $\gamma_{21}$ . Thus the reflection coefficient of the discontinuity for incidence from the right is proportional to  $\gamma_{21} k^{-J-1}$ .

To determine  $\gamma_{21}$  we could use (33). However, we shall instead analyze (2) directly, without transforming it into a first order system because the direct analysis is usually simpler for a single equation. Therefore, our present calculations will be independent of the preceding ones. We shall seek two linearly

independent solutions  $u_1(x, k)$  and  $u_2(x, k) = \bar{u}_1(x, k)$ , the complex conjugate of  $u_1$ . Our analysis will show that  $u_2$  has the asymptotic form

$$(39) \quad u_2(x, k) = \exp \left[ -ik \int_{x_0}^x n dx \right] \left[ \sum_{j=0}^{J+1} v_j(x) (-ik)^{-j} + o(k^{-J-1}) \right] \quad x \leq x_0$$

$$(40) \quad u_2(x, k) = \exp \left[ -ik \int_{x_0}^x n dx \right] \left[ \sum_{j=0}^{J+1} v_j(x) (-ik)^{-j} + o(k^{-J-1}) \right] \\ + \exp \left[ ik \int_{x_0}^x n dx \right] \left[ \gamma v_0(x) (-ik)^{-J-1} + o(k^{-J-1}) \right] \quad x \geq x_0$$

Upon inserting (39) or (40) into (2) and equating to zero the coefficient of each power of  $k$ , we obtain

$$(41) \quad 2n v'_j + n' v_j = -v''_{j-1} \quad j = 0, \dots, J+1$$

The continuity of  $u_2$  at  $x_0$  yields

$$(42) \quad [v_j] = 0 \quad j = 0, \dots, J$$

$$(43) \quad \gamma = -[v_{J+1}]/v_0(x_0)$$

Continuity of  $u'_2$  at  $x_0$  yields

$$(44) \quad [v_j] = -[v'_{j-1}]/n(x_0) \quad j = 0, \dots, J$$

$$(45) \quad n(x_0)[v_{J+1}] = \gamma n(x_0)v_0(x_0) - [v'_J]$$

From (42) and (44) we find

$$(46) \quad [v'_j] = 0 \quad j = 0, \dots, J-1$$

From (43) and (45) we obtain

$$(47) \quad \gamma = [v'_J]/2v_0(x_0)n(x_0)$$

To calculate  $\gamma$  we first note that we can find solutions of (41) such that  $v_j$  has  $J - j$  continuous derivatives, for  $j = 0, \dots, J$ . Then (42) and (46) are satisfied. Then we differentiate (41)  $m$  times and subtract the result at  $x_0^-$  from that at  $x_0^+$  obtaining

$$(48) \quad 2n(x_0) [v_j^{(m)}] + v_j(x_0) [n^{(m)}] = - [v_{j-1}^{(m+1)}], \quad m=1, \dots, J-j+1; j=0, \dots, J$$

Upon setting  $m = J-j+1$  in (48), we obtain for  $j \neq 0$ ,

$$(49) \quad [v_j^{(J-j+1)}] = - [v_{j-1}^{(J-j+2)}] / 2n(x_0), \quad j=1, \dots, J$$

If we set  $j = 0$  and  $m = J+1$  in (48) we get

$$(50) \quad [v_0^{(J+1)}] = - [n^{(J+1)}] v_0(x_0) / 2n(x_0)$$

Now we set  $j = J+1$  in (49), and then using (49) repeatedly, we obtain

$$(51) \quad [v_{J+1}] = - [v_J'] / 2n(x_0) = \left( \frac{-1}{2n(x_0)} \right)^2 [v_{J-1}^{(2)}] \\ = \left( \frac{-1}{2n(x_0)} \right)^{J+2} v_0(x_0) [n^{(J+1)}]$$

Thus (43) or (47) yields

$$(52) \quad \gamma = - \left( \frac{-1}{2n(x_0)} \right)^{J+2} [n^{(J+1)}]$$

On the basis of these results, we can determine the reflection coefficient of the variable medium with index  $n(x)$ . For this purpose we must suppose that  $n(x)$  tends to the value one as  $x$  tends to  $\pm \infty$ . In fact,  $n(x)$  must tend to one fast enough so that  $n(x) - 1$  is integrable. We now observe that  $u_2$  represents a wave incident from  $+\infty$  which is partly transmitted and partly reflected at

$x = x_o$ . The amplitudes  $A_{inc}$  and  $A_{ref}$  of the incident and reflected waves are defined as

$$(53) \quad A_{inc} = \exp \left[ -ik \left( \int_{x_o}^{\infty} (n-1) dx - x_o \right) \right] v_o(\infty) ,$$

$$(54) \quad A_{ref} = \exp \left[ ik \left( \int_{x_o}^{\infty} (n-1) dx - x_o \right) \right] \gamma k^{-J-1} v_o(\infty) .$$

Then the reflection coefficient  $R_1$  is given by

$$(55) \quad R_1 = \frac{A_{ref}}{A_{inc}} = \gamma (-ik)^{-J-1} \exp \left[ 2ik \left( \int_{x_o}^{\infty} (n-1) dx - x_o \right) \right] \\ = \frac{1}{2n} \left( \frac{1}{2ikn(x_o)} \right)^{J+1} \left[ \frac{d^{J+1}n}{dx^{J+1}} \right] \exp \left[ 2ik \left( \int_{x_o}^{\infty} (n-1) dx - x_o \right) \right] .$$

To determine the reflection coefficient for waves incident from the left, we must construct a solution which contains only a transmitted wave for  $x > x_o$ . Such a solution is  $u_1 - \gamma(-ik)^{-J-1}u_2$ , at least through terms of order  $k^{-J-1}$ . From it we find

$$(56) \quad R_2 = -\gamma(-ik)^{-J-1} \exp \left[ -2ik \left( \int_{-\infty}^{x_o} (n-1) dx + x_o \right) \right] \\ = -\frac{1}{2n(x_o)} \left( \frac{1}{2ikn(x_o)} \right)^{J+1} \left[ \frac{d^{J+1}n}{dx^{J+1}} \right] \exp \left[ -2ik \left( \int_{-\infty}^{x_o} (n-1) dx + x_o \right) \right] .$$

5. The case of a discontinuous  $A_0$

Let us finally consider the case in which  $A_0$  is discontinuous, which was not included in the preceding sections. In this case the matrix  $R$  is not necessarily continuous, and therefore we do not introduce it. Instead we work directly with (1) and prove the following

Theorem III Suppose that in an interval  $I$  of the  $x$  axis,  $A(x, k)$  has for large  $k$  the asymptotic form

$$(57) \quad A(x, k) = A_0(x) + o(1).$$

Let  $A_0(x)$  have  $N$  distinct eigenvalues  $\lambda_i(x)$ ,  $i = 1, \dots, N$ , in  $I$  and let  $A_0$  be continuous in  $I$  except for a finite jump at  $x_0$ . Then there are  $N$  linearly independent solutions of (1),  $u_i(x, k)$ ,  $i = 1, \dots, N$ , with the following asymptotic forms for large  $k$

$$(58) \quad u_i(x, k) = \exp\left[k \int_{x_0}^x \lambda_i(x) dx\right] [u_i^0(x) + o(1)], \quad x \leq x_0$$

$$(59) \quad u_i(x, k) = \sum_{t=1}^N \exp\left[k \int_{x_0}^x \lambda_t(x) dx\right] [c_{it}^0 \tilde{u}_i^0(x) + o(1)], \quad x \geq x_0$$

To prove this result we first apply theorem I with  $J_0 = 0$  in the region  $x \leq x_0$  to establish the existence of  $N$  linearly independent solutions  $u_i$  for which (58) holds. Then we apply it again for  $x \geq x_0$  to obtain  $N$  other linearly independent solutions  $\tilde{u}_i$  which have there asymptotic forms of the type shown in (58) with coefficients  $\tilde{u}_i^0(x)$  in place of  $u_i^0(x)$ . Next we express  $u_i$  in terms of the  $\tilde{u}_i$  in the form

$$(60) \quad u_i(x, k) = \sum_{t=1}^N c_{it}(k) \tilde{u}_t(x, k).$$

We then apply (60) at  $x = x_0$  using the asymptotic form (58) for  $u_i$  and the corresponding one for  $\tilde{u}_t$ . The resulting equation shows that  $c_{it} = c_{it}^0 + o(1)$  where  $c_{it}^0$  is determined by the equations.

$$(61) \quad u_i^0(x_0) = \sum_{t=1}^N c_{it}^0 \tilde{u}_t^0(x_0), \quad i = 1, \dots, N.$$

The solvability of (61) is assured by the linear independence of the  $\tilde{u}_i(x)$ . When the asymptotic form of  $c_{it}$  and that of  $\tilde{u}_t(x)$  in the region  $x \leq x_0$  are inserted into (60), (59) results. This completes the proof of the theorem, which is essentially the extension of Theorem II to the previously excluded case  $J = -1$ . The results of (52), (55) and (56) of Section 4 also hold for  $J = -1$  if  $2n(x_0)$  in them is replaced by  $n(x_0+) + n(x_0-)$ .

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